Institut für Makroökonomie
und Konjunkturforschung
Macroeconomic Policy Institute

Working Paper

3/2011

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this version: February 2011

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The estimation uncertainty of permanent-transitory decompositions in cointegrated systems*

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Abstract

The topic of this paper is the estimation uncertainty of the Stock-Watson and Gonzalo-Granger permanent-transitory decompositions in the framework of the cointegrated vector-autoregression. Specifically, we suggest an approach to construct the confidence interval of the transitory component in a given period (e.g. the latest observation) by conditioning on the observed data in that period. To calculate asymptotically valid confidence intervals we use the delta method and two bootstrap variants. As an illustration we analyze the uncertainty of (US) output gap estimates in a system of output, consumption, and investment.

Keywords: transitory components, VECM, delta method, bootstrap

JEL codes: C32 (multiple time series), C15 (simulation methods), E32 (business cycles)

*Thanks to seminar participants at DIW Berlin and University Potsdam for helpful comments. The author acknowledges the hospitality of the CES at University Paris 1 Panthéon-Sorbonne CNRS, where parts of the research for this paper were carried out. The numeric calculations have been produced with gretl 1.9 (Cottrell and Lucchetti, 2010), the code is available on request or from the web page of the author (www.sven.schreiber.name).

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1 Introduction

In this paper we suggest an approach to assess the estimation uncertainty of two permanent-transitory (PT) decompositions estimated in a cointegrated VAR framework, namely of the Stock and Watson (1988, SW) and Gonzalo and Granger (1995, GG) methods.

There are many ways to decompose integrated multivariate time series into their unobserved permanent and transitory components, even if we restrict our attention to additive decompositions $y_t = y_t^{perma} + y_t^{trans}$ (where y_t is an n-dimensional time series). The most widespread methods are state-space models estimated with the Kalman filter algorithm on the one hand,[1] and decompositions based on cointegrated VARs on the other hand (vector error-correction models, VECM).[2] The leading examples of the VECM-based decompositions are the extraction of SW common trends and GG common factors with their corresponding transitory components. The state-space approach is a powerful and flexible tool which also has the advantage that the Kalman filter provides a way to assess the uncertainty surrounding the estimates of the (smoothed) states and therefore of the permanent component. However, the estimation of state-space models poses the typical problems of iterative numerical methods, namely that they may fail to converge, and that different arbitrary choices of initial values (for the unobserved states) sometimes also have a considerable impact on the results. In contrast, the SW and GG measures only rely on the available estimated quantities from the VECM via closed-form algebraic expressions, and therefore these measures may be more desirable in certain applications.[3] However, for VECM-based measures there has not existed a way to

[1]These are also known by the generic name of unobserved-components models (Harvey and Proietti, 2005) or sometimes "structural" time series models, see for example Harvey and Shephard (1993).

[2]Here we do not consider univariate filters or smoothers like the Hodrick-Prescott or Baxter-King filter. Of course they could also be trivially applied to multivariate time series on an element-by-element basis, but we restrict our attention to those methods that take the multivariate system linkages explicitly into account.

[3]See below for the detailed formulas. We acknowledge the fact that if over-identifying restrictions are placed on the cointegration and/or the adjustment coefficients, the estimation of the coin-

quantify the estimation uncertainty for a given period of interest (i.e., for a given data constellation). It has only been possible so far to assess the significance of the coefficients of the transitory decomposition in general for the whole sample (on average), by standard test procedures. Therefore the goal of this paper is to provide additional tools to quantify the uncertainty around the SW and GG decompositions.

The entire analysis will be conducted conditional on a fixed cointegration rank. This means that a certain degree of model uncertainty will not be captured by our confidence bands, if the true cointegration rank is not treated as known a priori. But such a conditional analysis is a standard approach to construct standard errors for VECM coefficients (including the implied impulse-response coefficients). Typical state-space models also share the characteristic that the dimension of the permanent component needs to be fixed before estimation.

After introducing the model framework and fixing some notation in the following section, we first analyze the uncertainty of the GG transitory component (section 3), and then present the analogous approach with respect to the SW component (section 4). In section 5 both approaches are applied to a three-variable dataset inspired by the influential work of King, Plosser, Stock, and Watson (1991), but updated to include current data. Section 6 summarizes.

2 Framework and assumptions

Consider a standard n-dimensional VAR with p lags:

$$y_t = A_1 y_{t-1} + ... + A_p y_{t-p} + \mu + \varepsilon_t, \qquad t = p, ..., T \tag{1}$$

where the innovations are white noise with covariance matrix Θ. We can reparameterize this system as a VECM:

tegrating relations itself becomes a difficult numerical maximization problem, too (see for example Boswijk and Doornik, 2004). However, once the VECM is estimated, applying the mentioned PT decompositions is possible with closed-form algebraic expressions.

$$\Delta y_t = \alpha\beta' y_{t-1} + \sum_{i=1}^{p-1} B_i \Delta y_{t-i} + \mu + \varepsilon_t \tag{2}$$

When cointegration is present, the long-run matrix $\alpha\beta' = -I + \sum_{i=1}^{p} A_i$ has reduced rank r which is the number of linearly independent cointegration relationships (and is also the column rank of the $n \times r$ matrices α and β). The coefficients of the lagged differences are given by $B_j = -\sum_{i=j+1}^{p} A_i$. We define the lag polynomial $B(L) = I - \sum_{i=1}^{p-1} B_i L^i$. Because it will be repeatedly needed below, we introduce an abbreviation for the following term: $B_{1\alpha\beta}^{-1} \equiv [B(1) - \alpha\beta']^{-1}$.

It is well known that the constant term μ can serve two purposes: if unrestricted, it may represent a linear drift term in the levels of the variables, as well as balancing the mean of the cointegrating relations. But if it is restricted as $\mu = \alpha\mu_0$, the levels of the data are assumed to be free of linear trend components.[4]

Apart from standard regularity conditions like a well-behaved distribution of the residuals ε_t (such that the standard asymptotic results for the VECM apply), we make the following assumptions:

Assumption 1. *All variables are individually $I(0)$ or $I(1)$.*

This assumption rules out higher integration orders.

Assumption 2. *Fixed cointegration rank r, $n > r > 0$.*

The cointegration rank may either be known, or its determination is treated as a pre-test outside of the estimation problem of the VECM and the transitory components of the data.

Assumption 3. *The cointegration coefficients β are estimated by maximum likelihood ("Johansen procedure") and are properly normalized and identified.*

[4] In the following, we will deal with the more general case of an unrestricted constant, which is much more popular in economics given the trending behavior of many variables in growing economies. As a further deterministic component it would also be possible for our analysis to allow a linear trend term in the cointegrating relations, because the convergence rate of its estimator is also greater than $\sqrt{T}$. (It may be advisable in practical work to normalize the trend term to have mean zero.) Our explicit formulation in this paper focuses on the presented case, however.

This assumption serves to achieve super-consistency of the estimates of the cointegration coefficients, see Paruolo (1997). Inter alia it means that identification is achieved by imposing restrictions on β, not on α, and that no coefficients with a true value of zero are "normalized" to a non-zero value. Other super-consistent estimation methods may be used as well.

For the application of the delta method as well as the bootstrap it is important to ascertain the asymptotic properties of the estimators. Formally, we collect the underlying coefficients of the model in one matrix $K = (\alpha, B_1, ..., B_{p-1}, \mu)$ and stack the coefficients in the vector $k = vec(K)$; this vector has $nr + n^2(p-1) + n$ elements that are freely varying.[5] Note that β is not included here because its estimate has to be treated as asymptotically fixed given its higher convergence rate (T instead of $\sqrt{T}$), i.e. its variation is asymptotically dominated by the variation of the estimators of the K elements. The OLS estimate of this vector k is asymptotically normally distributed and has $\sqrt{T}$-convergence.[6]

Lemma 1. *Standard asymptotics of the underlying coefficients:*

$$\sqrt{T}(\hat{k} - k) \to N(0, \Omega)$$

The covariance matrix Ω can be easily estimated within the standard system OLS estimation once the super-consistent estimate $\hat{\beta}$ has been determined.

Conditioning: The idea in this paper is to condition on the observed data at (or for the SW decomposition: around) an observation period τ. In order to determine the resulting *conditional* estimation uncertainty of the transitory components, we therefore must only consider the randomness stemming from the remaining observations. A simple way of doing this is to actually remove the conditioning data

[5]The only qualification here is given by the standard assumptions that were made about the model class, i.e. the cointegration rank must be preserved and the system must not become integrated of higher order. These requirements are fulfilled in the neighborhood of the true parameters.

[6]This also applies to the $\alpha_\perp$-directions of the constant term, see Paruolo (1997).

from the likelihood function; this could be achieved either by using impulse dummies for the corresponding observations, or in the often interesting case of the end of the sample, by simply shortening the sample. Alternatively ("lazily") we may adopt a weaker position, still use all data in the sample, and make only sure that the conditioning data do not affect the estimation *asymptotically*. For the Gonzalo-Granger decomposition this requirement is automatically fulfilled, since there the transitory component for a period τ only depends on the single contemporaneous observation (see below) which of course is asymptotically negligible. But for this "lazy" approach to work in the case of the SW decomposition –where lagged values are involved (see below)– we need to limit the extent of the conditioning data, for which we may use the following assumption.

Assumption 4. *The lag length is at most growing slowly.*

We must make sure that the conditioning data does not affect the estimates asymptotically, if we use all data in the sample for convenience. For the extraction of the Stock-Watson transitory components we need to assume e.g. $\lim_{T\to\infty} \frac{p}{\sqrt{T}} = 0$*. This is a sufficient condition, and it is obviously fulfilled for a fixed lag length* p*.*

3 The uncertainty of the Gonzalo-Granger decomposition

3.1 Definition and representation of the GG decomposition

As shown by Gonzalo and Granger (1995), when the permanent and transitory components are assumed to be linear combinations of the contemporaneous values y_t only, the PT decomposition is uniquely given as follows:

$$y_t = \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp y_t + \alpha(\beta'\alpha)^{-1}\beta' y_t, \qquad (3)$$

where the first part is the non-stationary permanent component, and the second part is the transitory component given by a linear combination of the cointegrating relationships.

We will use the alternative formulation by Hecq, Palm, and Urbain (2000) (based in turn on Proietti, 1997) which proves especially useful with the Stock-Watson decomposition below.[7] For that representation we need to define the following terms: First another lag polynomial is defined if $p > 1$: $B^*(L) = B_0^* + B_1^* L + \ldots + B_{p-2}^* L^{p-2}$, where $B_j^* = \sum_{i=j+1}^{p-1} B_i$. With this setup an important projection matrix is given by

$$P = B_{1\alpha\beta}^{-1} \alpha \left[\beta' B_{1\alpha\beta}^{-1} \alpha \right]^{-1} \beta' \tag{4}$$

Since $\psi_{1t} = P y_t$ is a linear combination of the cointegrating relations $\beta' y_t$ it is obviously stationary, and it is actually shown by Proietti (1997) that this is just the GG transitory component:

$$y_t^{transGG} = \psi_{1t} = P y_t \tag{5}$$

This transitory component will in general have a non-zero mean, however. For an economic interpretation it is especially useful to consider a transformation of the transitory component which will have an unconditional expectation of zero, because the sign of that transformed component automatically tells us whether the observed level of a variable is below or above its permanent component. For example the sign of an output gap estimate is important for identifying a recessionary or overheating economy.

To this end we use the expression (again adapted from Proietti, 1997) for the mean of the cointegrating relationships:

[7] The notation is not completely identical, however.

$$\mathsf{E}(\beta' y_t) = -\left(\beta' B_{1\alpha\beta}^{-1}\alpha\right)^{-1}\beta' B_{1\alpha\beta}^{-1}\mu, \tag{6}$$

which enables us to calculate the de-meaned transitory component:

$$\begin{aligned}
\tilde{\psi}_{1t} &= \psi_{1t} - \mathsf{E}(\psi_{1t}) \\
&= B_{1\alpha\beta}^{-1}\alpha[\beta' B_{1\alpha\beta}^{-1}\alpha]^{-1}\left(\beta' y_t + [\beta' B_{1\alpha\beta}^{-1}\alpha]^{-1}\beta' B_{1\alpha\beta}^{-1}\mu\right) \\
&= B_{1\alpha\beta}^{-1}\alpha[\beta' B_{1\alpha\beta}^{-1}\alpha]^{-1}\left(\beta'; [\beta' B_{1\alpha\beta}^{-1}\alpha]^{-1}\beta' B_{1\alpha\beta}^{-1}\mu\right)(y_t'; 1)' \qquad (7) \\
&\equiv G(y_t'; 1)'
\end{aligned}$$

Of course it is well known how to test the hypothesis that the GG transitory component of a certain variable vanishes completely. From the definitions of the GG decomposition it is clear that this involves a test that the i-th row of α is zero, which is a standard test problem given the cointegration rank and the estimated cointegration coefficients. This paper is instead concerned with the uncertainty of the transitory component at a certain period, assuming that it exists at all.

3.2 The Delta method for the GG decomposition

We can express the de-meaned transitory GG component $\tilde{\psi}_{1\tau}$ in period $\tau \in \{p, ..., T\}$ as a function of the underlying short-run $\sqrt{T}$-consistent coefficient vector k, of the super-consistent cointegration coefficients β, and of the data; since the Gonzalo-Granger transitory component $\tilde{\psi}_{1\tau}$ only depends on the contemporaneous observations, we only need to condition on y_τ:

$$\tilde{\psi}_{1\tau} = f_{GG}(k; \beta, y_\tau) \tag{8}$$

The function f_{GG} is of course given by the derivations of the transitory components above. Let J_{GG} be the Jacobian matrix of that function with respect to k, treating the cointegration coefficients β as (asymptotically) fixed and conditioning on the data in period τ. In our illustration below, we use a numerical approximation to the Jacobian as implemented by the fdjac() function in gretl. With this definition we can state the first result with respect to the estimation uncertainty of the GG transitory component.

Proposition 1. *The conditional asymptotic distribution of the GG transitory component estimator for a fixed y_τ is given by:*

$$\sqrt{T}(\hat{\tilde{\psi}}_{1\tau} - \tilde{\psi}_{1\tau}) \quad \rightarrow \quad N(0, J_{GG}\Omega J_{GG}^{'}), \tag{9}$$

Proof. The proposition follows directly as an application of the standard delta method, where Ω is the covariance matrix of the underlying coefficients in k, defined as before. Given the T-convergence of the cointegration coefficient estimates $\hat{\beta}$, their variation is asymptotically dominated by that of the other coefficients and thus formally negligible. The influence of y_τ on the estimates is either non-existent (if a dummy variable for period τ was used) or asymptotically negligible. A standard system OLS estimate $\hat{\Omega}$ (for a given $\hat{\beta}$) can be used for a feasible version of this proposition. □

The practical drawback of this formulation is that it would have to be re-calculated for every τ. However, we can use the fact that the conditioning data are just a post-multiplied factor. So if we are interested in the i-th element of the transitory component we can use the i-th row of G: $\tilde{\psi}_{1\tau,i} = g_i^{'}(y_\tau'; 1)'$. This row is obviously also a function of the underlying coefficients, but not of the data: $g_i = f_{g,i}(k; \beta)$. We can denote the Jacobian of the function $f_{g,i}$ by $J_{g,i}$, and then express the variance of the estimated i-th transitory component directly as in the following corollary.

Corollary 1. *The variance of the GG transitory component estimator for a certain* *variable in a certain period is given by:*

$$\mathsf{Var}(\hat{\hat{\psi}}_{1\tau,i}) = (y'_\tau;1)J_{g,i}\Omega J'_{g,i}(y'_\tau;1)' \tag{10}$$

Since $J_{g,i}$ is not a function of the data we only have to perform between 1 and n Jacobian computations (depending on how many variables we are interested in) instead of $T-p$. Nevertheless, it is important to keep in mind that the derived confidence intervals are only valid for the chosen period τ and not as confidence bands for the entire sample, since we cannot condition on the entire sample and still have random estimates. When we display our calculations in a form that resembles confidence bands for the time series, it is just done for convenience, since different readers may be interested in different periods.

3.3 The bootstrap method

The justification of the bootstrap in this case rests essentially on the same foundations as the delta method before. The underlying coefficients are freely varying (for a maintained cointegration rank r), and the asymptotic distribution of the transitory components conditional on the data at a certain observation period τ is well-behaved. Of course we hope that the bootstrap may yield some small-sample refinements over the asymptotic approximation by the delta method, for example by taking into account explicitly the variation of the cointegration coefficients estimates.

To be concrete, the distribution for the period of interest τ can be simulated with the following algorithm. As a starting point we can use the standard estimates of (2) that we already used for the delta method.

1. Using the point estimates as the auxiliary data-generating process, simulate artificial data for the periods $t = p...T$ by drawing from a suitable distribution describing the innovation process ε_t. This could either be a random draw

from a fitted parametric distribution like a multivariate normal distribution with covariance matrix $\widehat{\Theta}$ (and mean zero, of course), or resampling from the estimated residuals. We will use the observed values of y_t as the initial values of the artificial data in periods $t = 0..p-1$. The resulting artificial data may be very different from the original data because it will have different underlying realizations of the stochastic trends, but the coefficients of the model will be comparable.

2. Re-estimate the VECM using the same specification that was applied to the original data, but with the artificial data created in the previous step. Then record the estimates of $\tilde{\psi}_{1\tau}$ as defined in equation (7), which means using the new estimated G coefficients of the current simulation run, but always employing the originally observed data $(y'_\tau; 1)$. Denote that estimate by $\tilde{\psi}_{1\tau,w}$, where w is a simulation index running from 1 to some sufficiently large integer W.

3. Repeat the previous two steps W times to get simulated distributions of (the estimate of) $\tilde{\psi}_{1\tau}$.

4. For the i-th variable calculate variants of the confidence intervals for the estimate of $\tilde{\psi}_{1\tau}$ in the following two ways:

   (a) First we base the intervals directly on the distributions of $\tilde{\psi}_{1\tau,w}$ over all w and construct a confidence interval using the empirical quantiles of the simulated distributions: with γ as the nominal coverage of the error band (1 minus the type-1 error) and the quantiles of $\tilde{\psi}_{1\tau,w}$ given by $\tilde{\psi}_{1\tau,(1-\gamma)/2}$ and $\tilde{\psi}_{1\tau,(1+\gamma)/2}$, the intervals are constructed as

$$[\tilde{\psi}_{1\tau,(1-\gamma)/2}; \tilde{\psi}_{1\tau,(1+\gamma)/2}]. \tag{11}$$

   This construction is analogous to what Sims and Zha (1999) have called “other-percentile” bands in the slightly different context of impulse-response analysis, and they criticized their use as “clearly [amplifying] any bias present in the estimation procedure” (p. 1125).

(b) Because of this criticism we also consider a Hall-type bootstrap, where the relevant distributions are given by $\tilde{\psi}_{1\tau,w} - \tilde{\psi}_{1\tau}$, i.e., for each variable and period the bootstrap realizations are corrected by the original point estimate.[8] Denoting the quantiles of these corrected distributions by $(\tilde{\psi}_{1\tau,w} - \tilde{\psi}_{1\tau})_{(1-\gamma)/2}$ and $(\tilde{\psi}_{1\tau,w} - \tilde{\psi}_{1\tau})_{(1+\gamma)/2}$, the Hall-type error bands are given by

$$[\tilde{\psi}_{1\tau} - (\tilde{\psi}_{1\tau,w} - \tilde{\psi}_{1\tau})_{(1+\gamma)/2}; \tilde{\psi}_{1\tau} - (\tilde{\psi}_{1\tau,w} - \tilde{\psi}_{1\tau})_{(1-\gamma)/2}]. \tag{12}$$

Note that the upper quantiles of the corrected distributions are used for the calculation of the lower error band margins, and vice versa. This “counter-acting swapping” serves to cancel out any bias of the estimation procedure.

Of course the bootstrap procedure can be simultaneously applied to all periods in the sample. However, we still do not get confidence “bands” because we cannot condition on the entire sample and do valid inference. As with the delta method, we can only derive valid confidence intervals for certain periods of interest.

[8] In order not to overload the notation, we do not formally distinguish here between the true transitory component (true of course conditional on period-τ data) and its original point estimate, because we hope it is clear from the context that only the estimate can be used here.

4 The uncertainty of the Stock-Watson decomposition

4.1 Definition and representation of the SW decomposition

In a standard formulation, and assuming a fixed initial value, the permanent SW components are given by

$$y_t^{permaSW} = y_0 + C\mu t + C\sum_{s=1}^{t} \varepsilon_s, \tag{13}$$

where C is the long-run moving-average impact matrix of reduced rank (which however is not directly of interest here). For the cointegrated VAR model the SW decomposition essentially yields the multivariate Beveridge-Nelson decomposition, i.e. the permanent component is a multivariate random walk. In contrast, the permanent component of the GG decomposition is autocorrelated in differences. This property of the SW decomposition implies an appealing interpretation: Given our knowledge at time t, only the SW transitory component of the time series is expected to change in the future (because it is expected to converge to its unconditional expectation, or in the demeaned case, to zero), so it is especially important for forecasting. Of course, the GG and SW permanent components only differ by stationary terms and are cointegrated, therefore they share the same long-run features.

Again following Proietti (1997) and Hecq, Palm, and Urbain (2000) the transitory SW component can be written as the sum of two terms,

$$y_t^{transSW} = \psi_{1t} + \psi_{2t}, \tag{14}$$

where the part ψ_{1t} represents the error-correcting movements of the system and is identical to the GG transitory component above, while the part ψ_{2t} are the remaining transitory movements of the system which do not contribute to the long-run equilibrium. This latter part can be represented as a distributed lag of the observable variables:

$$\psi_{2t} = -(I-P)B_{1\alpha\beta}^{-1}B^*(L)\Delta y_t \tag{15}$$

This second part remains to be demeaned as well, which can be achieved by using the known unconditional expectation of the differences:

$$\mathsf{E}(\Delta y_t) = (I-P)B_{1\alpha\beta}^{-1}\mu \tag{16}$$

Using the abbreviation $\mu^* \equiv (I-P)B_{1\alpha\beta}^{-1}\mu$ we can now write:

$$\begin{aligned}
\tilde{\psi}_{2t} &= \psi_{2t} - \mathsf{E}(\psi_{2t}) \\
&= -(I-P)B_{1\alpha\beta}^{-1}B^*(L)\left(\Delta y_t - \mu^*\right) \\
&= \left(-\left[I-P\right]B_{1\alpha\beta}^{-1}\right)\left(B_0^*;-B_0^*\mu^*;B_1^*;-B_1^*\mu^*;...;B_{p-2}^*;-B_{p-2}^*\mu^*\right)\times \\
&\quad (\Delta y_t';1;\Delta y_{t-1}';1;\ldots;\Delta y_{t-p+2}';1)' \qquad (17)\\
&= \left(-\left[I-P\right]B_{1\alpha\beta}^{-1}\right)\left(B_0^*;B_1^*;...;B_{p-2}^*;-B^*(1)\mu^*\right)\times \\
&\quad (\Delta y_t';\Delta y_{t-1}';\ldots;\Delta y_{t-p+2}';1)' \\
&= S_1S_2\left(\Delta y_t;\Delta y_{t-1};\cdots;\Delta y_{t-p+2};1\right)'
\end{aligned}$$

Then combining the two parts we have for the SW transitory component:

$$\begin{aligned}
\tilde{\psi}_t &= \tilde{\psi}_{1t} + \tilde{\psi}_{2t} \\
&= \left(P;-\left[I-P\right]B_{1\alpha\beta}^{-1}\left[B_0^*;B_1^*;...;B_{p-2}^*\right];s_\mu\right)\times \qquad (18)\\
&\quad (\Delta y_t';\Delta y_{t-1}';\ldots;\Delta y_{t-p+2}';1)' \\
&\equiv S(\Delta y_t';\Delta y_{t-1}';\ldots;\Delta y_{t-p+2}';1)',
\end{aligned}$$

where the last element relating to the constant term is given by

$$s_\mu = \left(B_{1\alpha\beta}^{-1}\alpha[\beta' B_{1\alpha\beta}^{-1}\alpha]^{-1}[\beta' B_{1\alpha\beta}^{-1}\alpha]^{-1}\beta' + [I-P]B_{1\alpha\beta}^{-1}B^*(1)\,[I-P]\right)B_{1\alpha\beta}^{-1}\mu. \tag{19}$$

Note that also for the SW transitory component it is known how to test the hypothesis that it vanishes for a certain variable. In addition to the zero row of α that was needed for the vanishing GG component, here the i-th rows of the various short-run coefficient matrices would also have to be zero. These restrictions essentially mean that the variable would be a strongly exogenous random walk. Again, for a given cointegration rank and super-consistently estimated cointegration coefficients, this is a standard test problem.

4.2 The Delta method for the SW decomposition

The calculation of the uncertainty for the SW transitory component is analogous to the procedure for the GG component above. Again we can express the $\tilde{\psi}_\tau$ (the demeaned overall transitory components) in period $\tau \in \{p,...,T\}$ as a function of k, of the cointegration coefficients β, and of the data; the only difference now is that we have to condition on the lagged values as well, $y_\tau,...,y_{\tau-p+1}$:

$$\tilde{\psi}_\tau = f_{SW}(k;\beta,y_\tau,...,y_{\tau-p+1}) \tag{20}$$

The function f_{SW} is again given by the derivations of the the transitory components above. Let J_{SW} be the Jacobian matrix of that function. We can state the estimation uncertainty of $\tilde{\psi}_\tau$ similar to the one of the GG decomposition in proposition 1.

Proposition 2. *The conditional asymptotic distribution of the SW transitory com-*

ponent estimator is given by:

$$\sqrt{T}(\hat{\tilde{\psi}}_\tau - \tilde{\psi}_\tau) \;\; \rightarrow \;\; N(0, J_{SW}\Omega J_{SW}^{'}) \tag{21}$$

Proof. Again the result follows directly from applying the delta method, cf. the remarks on proposition 1, where now the conditioning data are given by $y_\tau, ..., y_{\tau-p+1}$. The influence of these data on the actual estimates is either non-existent (if appropriate dummies have been used in estimating the system), or under assumption 4 it is asymptotically vanishing. □

As before, we can use the fact that the conditioning data are just a linear factor post-multiplied to S. Using the abbreviation $y'_{cond,t} = (\Delta y'_t; \Delta y'_{t-1}; \ldots; \Delta y'_{t-p+2}; 1)$, if we are interested in the i-th element of the transitory component we can use the i-th row of S: $\tilde{\psi}_{\tau,i} = s_i^{'} y_{cond,\tau}$. This row is obviously also a function of the underlying coefficients, but not of the data: $s_i = f_{s,i}(k;\beta)$. We define the Jacobian of the function $f_{s,i}$ as $J_{s,i}$, and like in the GG case we directly express the variance of the estimated i-th transitory component in an analogous corollary.

Corollary 2. *The variance of the SW transitory component estimator for a certain variable in a certain period is given by:*

$$\mathsf{Var}(\hat{\tilde{\psi}}_{\tau,i}) = y_{cond,\tau}^{'} J_{s,i} \Omega J_{s,i}^{'} y_{cond,\tau} \tag{22}$$

Again, since $J_{s,i}$ is not a function of the data we only have to perform between 1 and n Jacobian computations (depending on how many variables we are interested in) instead of $T - p$. Nevertheless this approach is still just a computational convenience device, because the interpretation remains only valid for a single chosen period.

4.3 The bootstrap method for the SW component

The bootstrap method in this case is completely analogous to the GG case and to save space we will not repeat the details of the algorithm here. Essentially, the distribution of the G coefficients is replaced by that of the S coefficients, and of course the transitory component must be constructed using the extended data vector $y_{cond,t}$ which includes lags, according to the formulas in section 4.1.

5 Illustration

For an illustration of how the methods work in practice we use a three-variable dataset inspired by the influential King, Plosser, Stock, and Watson (1991, KPSW) article dealing with stochastic trends in US business-cycle analysis. That is, we also use the quarterly variables (logs of) real consumption $cons_t$, real (gross) investment inv_t, and real output inc_t, but instead of their sample 1947-1988 we analyze more recent data spanning 1968q1-2010q2. We also let the series be tied together by two cointegrating relationships ($r = 2$), such that any two of the three variables are cointegrated. KPSW propose to specify the cointegrating relationships according to economic theory as the "great ratios" of balanced growth, specifically $cons - inc$ and $inv - inc$, but for the purposes of this illustration we will work with freely estimated cointegration coefficients β.[9] No exogenous terms are included in the cointegration space, and the constant term is unrestricted to account for the deterministic long-run growth trend. The standard choice of four lags ($p = 4$) for quarterly data is also reasonable here.

For the transitory components we focus on the output gap; figure 1 shows the point estimates of the transitory components of both decompositions, GG and SW. Both estimates seem quite similar for this data –apart from fluctuations of the SW

[9] These great ratios are actually not so great in terms of their stationarity properties in the sub-sample after the publication of KPSW. This is another reason to freely estimate the cointegration coefficients instead of imposing unit values.

gap measure in the very short run– which may suggest the presence of common cyclical features (Proietti, 1997). The great recession of 2008-2010 is clearly visible as a large drop in the output gap. In general we note that a falling output gap measure (equivalent to a rising output gap in economic terms, since a positive measure indicates excess output) corresponds quite well to the NBER dating of the US recessions.

Before turning to the estimation uncertainty of the transitory components in specific periods it may be useful to briefly test whether the transitory output component is at all significant in this system. For the GG transitory component this check can be directly implemented as the standard test of the null hypothesis of a zero row in α for the output equation. In our illustration here, this test yields the following result: P(χ^2(2)>11.19) = 0.0037, and thus the GG output gap is clearly significant in general, over the entire sample. Obviously, this finding automatically implies the significance of the SW output gap, since having a row of zeroes in α is also a necessary (but not sufficient) condition for a vanishing SW transitory component.

In the next step we calculate the confidence intervals for the output gap as given by the GG decomposition, shown in figure 2. All intervals have a nominal asymptotic coverage of 90%, and we have employed the described computational shortcut where the observation on which we condition is still included in the estimation sample, but its influence should be negligible compared to the rest of the sample. For the bootstraps we resample from the estimated residuals.

First of all we notice that for most periods the intervals are quite similar. However, there are some exceptions; around 2005 for example the delta method intervals are tighter than their bootstrapped counterparts. And in the final observations for 2010 the Hall-type bootstrap intervals are shifted upwards somewhat in comparison to the naive bootstrap intervals (as well as compared to the delta method intervals, which are of course symmetric around the respective point estimates of the gap). Finally, whether the confidence intervals in general should be perceived as relatively

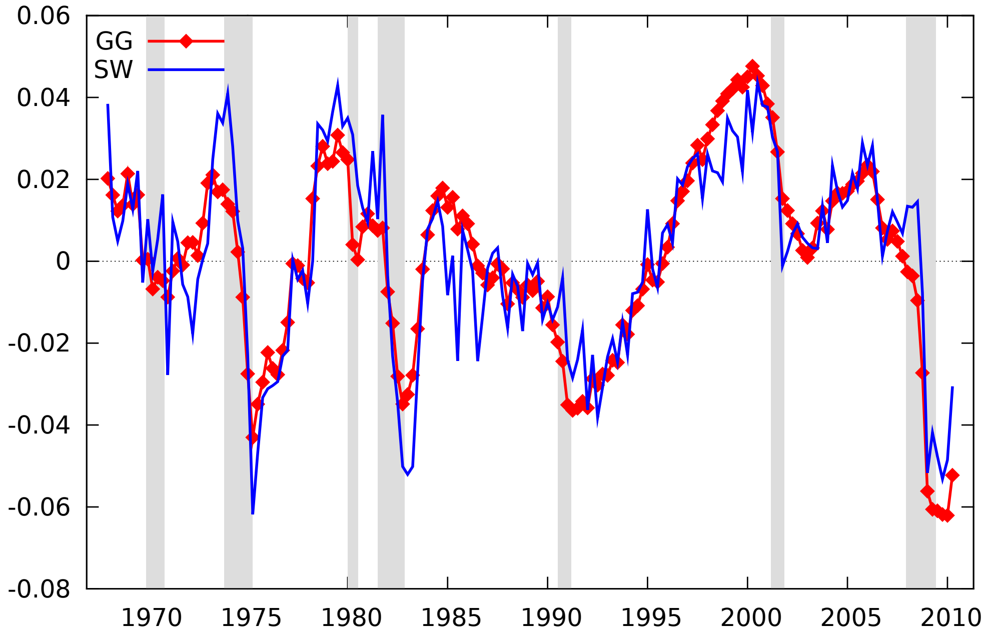

Figure 1: Estimated output gaps as transitory components of the GG and SW permanent-transitory decompositions. Shaded areas indicate NBER recession dating.

wide or tight is probably a matter of taste. Nevertheless, while for the latest observation(s) the interval is quite wide indeed, the output gap is still highly significantly different from zero.

Finally, figure 3 displays the corresponding measures and calculations for the SW decomposition of the output series in the cointegrated system. Similar remarks as before apply concerning the comparison of the three different interval "series". The most interesting difference with respect to the GG-based graph relates to the latest observation (2010q2): Given the considerably lower point estimate of the SW-based output gap (in absolute value) together with a comparably wide confidence interval, the latest observed output gap is not significantly different from zero under the SW decomposition.

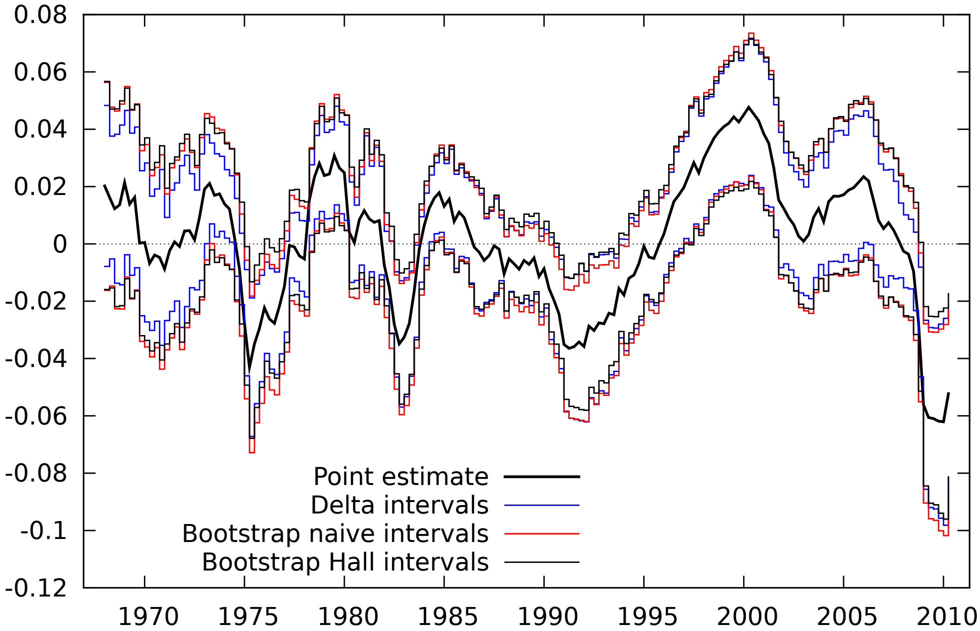

Figure 2: GG decomposition, confidence intervals for the output gap; displayed together for all periods in the sample for convenience, while the interpretation should be for a single period only.

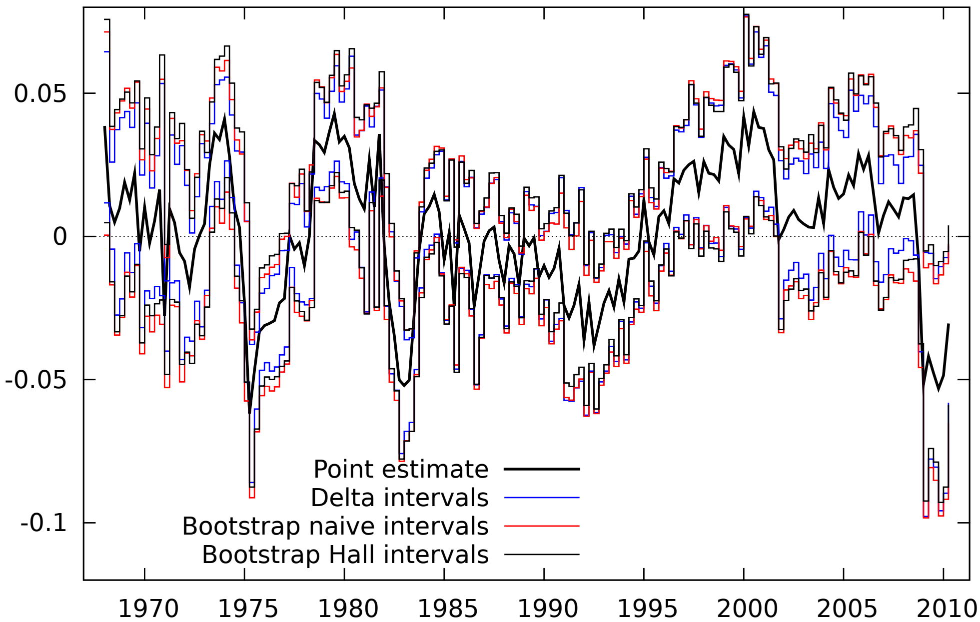

Figure 3: SW decomposition, confidence intervals for the output gap; displayed together for all periods in the sample for convenience, while the interpretation should be for a single period only.

6 Summary

While a permanent-transitory decomposition of non-stationary time series in a cointegrated system can always be mechanically calculated, it is not a priori clear if the resulting transitory component for a period of interest is significantly different from zero, given the sampling uncertainty of the estimated coefficients. So far it has only been possible to test the overall significance of the transitory components for the entire sample. In that sense even the sign of the transitory component in the period of interest cannot be fully established, which may be problematic for many economic applications.

Therefore, we have proposed an additional approach to assess the sampling uncertainty of widespread permanent-transitory decompositions, where we take as given the data constellations that are observed at the period of interest (possibly the latest observation period available). These measures provide additional information compared to the standard overall test results. For this conditional approach we have derived one delta-method and two bootstrap-based ways to quantify the estimation uncertainty of the Stock-Watson (common-trends-based) and Gonzalo-Granger (common-factor-based) decompositions.

In the empirical illustration we calculated the uncertainty of output gap estimates for the US. For example, at the 10% significance level (90% nominal coverage of the confidence intervals) it turned out that for the latest available observation (2010q2) the GG-based output gap is significantly different from zero, whereas the SW-based gap estimate is not.

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IMK Working Paper is an online publication series available at:
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ISSN: 1861-2199

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